LINEAR ALGEBRA HOMEWORK

JULY 27, 2023

Let F be a field and V a subspace of F^k for some $k \in \mathbb{N}$. Let A be an $m \times n$ matrix for some $m, n \in \mathbb{N}$ and b a vector in F^n. Denote by $V_b = \mathrm{Sol}(Ax = b)$ and $V_0 = \mathrm{Sol}(Ax = 0)$.

Exercise 1. ***Show*** *that:*

(1) *If $v_1, v_2 \in V$ are independent and $Fv_1 + Fv_2 \subsetneqq V$, then one can find $v_3 \in V \backslash (Fv_1 + Fv_2)$ such that v_1, v_2, v_3 are independent.*
(2) *More generally, if $v_1, \ldots, v_{r-1} \in V$ are independent and $Fv_1 + \cdots + Fv_{r-1} \subsetneq V$, then one can find $v_r \in V \backslash (Fv_1 + \cdots + Fv_{r-1})$ such that $v_1, \ldots, v_r$ are independent.*

Exercise 2. *Let $P \in V_b$.* ***Show*** *that $V_b = V_0 + P$.*

Exercise 3. *Define the sum on V_b as*

$$\begin{aligned} \oplus : & V_b \times V_b \longrightarrow V_b \\ & (y,\ y') \ \longmapsto y \oplus y' := y + y' - P. \end{aligned}$$

Please ***give*** *an appropriate definition on scalar-multiplication $\otimes$ and* ***show*** *that $(V_b, \oplus, \otimes, P)$ satisfies the algebra laws V1-V8.*